Elasticity Considerations for Optimal Pricing of Networks

Murat Yuksel[†], Shivkumar Kalyanaraman
Rensselaer Polytechnic Institute,
ECSE Department, JEC-6046
110 8th Street, Troy, NY, 12180, USA.
yuksem@rpi.edu, shivkuma@ecse.rpi.edu

Abstract

Static optimization of networks by pricing has attracted significant attention over the last decade. These studies assumed concave utility functions for users and derived optimal pricing strategies for the network provider. In this paper, we consider effect of user's elasticity to price and bandwidth on optimality of pricing. We first derive optimal pricing strategy for the case of logarithmic user utilities. Then, we investigate two types of elasticity for users: Demand-price elasticity and utility-bandwidth elasticity. By incorporating these two elasticities, we develop a non-logarithmic utility function for users. Finally, we derive an optimal pricing strategy for the non-logarithmic user utilities and illustrate that pricing strategy should be more conservative when the elasticities increase.

Index Terms

Static Optimization, Network Pricing, Congestion Pricing, Elasticity

I. Introduction

Recently, optimization of networks by pricing has been researched extensively [1], [2], [3], [4]. In [1], Kelly laid out overall optimization problem for a network, that is maximizing total user utility. He splitted the overall system problem into sub-problems of surplus maximization for the user and revenue maximization for the provider. He showed that network service prices can be used as Lagrange multipliers between the user's and the provider's problems. Then in [2], Kelly et al. provided centralized and decentralized pricing algorithms that will converge the system to the optimal, i.e. maximization of user utilities. Kelly and his co-workers, in their analysis, used logarithmic utility functions for

This work is sponsored by NSF under contract number ANI9819112.
[†]Corresponding author. Tel: +1-518-2766823

users. They proved that optimal rate allocation will be weighted proportional fair when users have utility function of form $u(x) = wlog(x)$, where w is weight and x is sending rate of the user.

Later in [3], Low et al. generalized the concepts to users with concave utility functions not necessarily logarithmic. They provided a family of distributed pricing algorithms that optimizes the network with users having logarithmic or non-logarithmic concave utilities.

One interesting issue that has not been investigated is the effect of user's elasticity on pricing strategies. In this paper, we investigate this particular issue by providing analytical arguments. We define two types of elasticity for users: Utility-bandwidth elasticity and demand-price elasticity. The latter one is the well-known elasticity in economics, i.e. users' demand elasticity to price. The former one is a new type of elasticity we define for users, i.e. user's utility elasticity to bandwidth. We formulate these two elasticities and demonstrate analytical association between them. We illustrate that pricing strategy must be more conservative in network resources (particularly capacity) when any of the two user elasticities increase.

The paper is organized as follows: First in Section II, we define the optimization problem of total user utility maximization and split it into two sub-problems by following Kelly's [1] ideas. Next in Section III, we solve the sub-problems for the case of logarithmic user utility functions, and derive optimal prices. Then we define utility-bandwidth elasticity and its relationship to demand-price elasticity in Section IV. In Section V, based on the elasticity definitions in Section IV, we define a general non-logarithmic utility function which includes the elasticities as parameters. We re-solve the optimization problem based on this new utility function, and derive the optimal prices again. Finally, we summarize the work in Section VI.

II. Problem Formulation

We now formulate the problem of *total user utility maximization* for a multi-user multi-bottleneck network.

Let $F = \{1, ..., F\}$ be the set of flows and $L = \{1, ..., L\}$ be the set of links in the network. Also, let $L(f)$ be the set of links the flow f passes through and $F(l)$ be the set of flows passing through the link l. Let c_l be the capacity of link l. Let λ be the vector of flow rates and λ_f be the rate of flow f. We can formulate the total user utility maximization problem as follows:

$SYSTEM$:

$$\max_{\lambda} \sum_{f} U_f(\lambda_f)$$

subject to

$$\sum_{f \in F(l)} \lambda_f \leq c_l, \quad l = 1, ..., L \tag{1}$$

This problem can be divided into two separate problems by employing monetary exchange between user flows and the network provider. Following Kelly's [1] methodology we split the system problem into two:

The first problem is solved at the user side. Given accumulation of link prices on the flow f's route, p^f, what is the optimal sending rate in order to *maximize surplus.*

$FLOW_f(p^f)$:

$$\max_{\lambda_f} \left\{ U_f(\lambda_f) - \sum_{l \in L(f)} p_l \lambda_f \right\}$$

over

$$\lambda_f \geq 0 \tag{2}$$

The second problem is solved at the provider's side. Given sending rate of user flows (which are dependent on the link prices), what is the optimal price to advertise in order to *maximize revenue.*

$NETWORK(\lambda(p^f))$:

$$\max_{p} \sum_{f} \sum_{l \in L(f)} p_l \lambda_f$$

subject to

$$\sum_{f \in F(l)} \lambda_f \ \leq \ c_l, \quad l = 1, ..., L$$

over

$$p \geq 0 \tag{3}$$

Let the total price paid by flow f be $p^f = \sum_{l \in L(f)} p_l$. Then, solution to $FLOW_f(p^f)$ will be:

$$U'_f(\lambda_f) = p^f$$

$$\lambda_f(p^f) = U_f'^{-1}(p^f) \tag{4}$$

When it comes to the $NETWORK(\lambda(p^f))$ problem, the solution will be dependent on user flows utility functions since their sending rate is based on their utility functions as shown in the solution of $FLOW_f(p^f)$. So, in the following sections we will solve the $NETWORK(\lambda(p^f))$ problem for the cases of logarithmic and non-logarithmic utility functions.

III. Optimal Prices: Logarithmic Utility Functions

We model customer i's utility with the well-known function[1] [2], [4], [6], [3]

$$u_i(x) = w_i log(x) \tag{5}$$

where x is the allocated bandwidth to the customer and w_i is customer i's budget (or bandwidth sensitivity).

Now, we set up a vectorized notation, then solve the revenue maximization problem $NETWORK(\lambda(p^f))$ described in the previous section. Assume the network includes n flows and m links. Let λ be row vector of the flow rates (λ_f for $f \in F$), P be column vector of the price at each link (p_l for $l \in L$). Define the $n \times n$ matrix P^* in which the diagonal element P^*_{jj} is the aggregate price being advertised to flow j (i.e. $p^j = \sum_{l \in L(j)} p_l$) and all the other elements are 0. Also, let A be the $n \times m$ routing matrix in which the element A_{ij} is 1 if ith flow is passing though jth link and the element A_{ij} is 0, if not, C be the column vector of link capacities (c_l for $l \in L$). Finally, define the $n \times n$ matrix $\hat{\lambda}$ in which the diagonal element $\hat{\lambda}_{jj}$ is the rate of flow j (i.e. $\hat{\lambda}_{jj} = \lambda_j$) and all the other elements are 0.

Given the above notation, relationship between the link price vector P and the flow aggregate price matrix P^* can be written as:

$$AP = P^* e \tag{6}$$

$$\lambda = (\hat{\lambda} e)^T = e^T \hat{\lambda}$$

where e is the column unit vector.

We use the utility function of (5) in our analysis. By plugging (5) in (4) we obtain flow's demand function in vectorized notation:

$$\lambda(P^*) = W P^{*-1} \tag{7}$$

where W is row vector of the weights w_i in flow's utility function (5). Similarly, we can write derivative of (7) as:

$$\lambda'(P^*) = -W(P^{*2})^{-1} \tag{8}$$

Also, we can write the utility function (5) and its derivative in vectorized notation as follows:

$$U(\lambda) = W log(\hat{\lambda}) \tag{9}$$

$$U'(\lambda) = W \hat{\lambda}^{-1} \tag{10}$$

The revenue maximization of (3) can be re-written as follows:

$$\max_P R = \lambda A P$$

[1] Wang and Schulzrinne introduced a more complex version in [5]. But the solutions we provide will be mainly similar even though we are using a simpler version of the function.

subject to

$$\lambda A \;\; \leq \;\; C^T. \tag{11}$$

So, we write the Lagrangian as follows:

$$L = \lambda AP + (C^T - \lambda A)\gamma \tag{12}$$

where γ is column vector of the Lagrange multipliers for the link capacity constrain.

By plugging (7) and (8) in appropriate places, the optimality conditions for (12) can be written as:

$$L_\gamma : C^T - WP^{*-1}A = 0 \tag{13}$$

$$L_{P^*} : -W(P^{*2})^{-1}P^*e + WP^{*-1}e - W(P^{*2})^{-1}A\gamma = 0 \tag{14}$$

By solving (14) for P^*, we obtain:

$$P^* = 0 \tag{15}$$

Now, solve (13) for P^*:

$$P^* = A(C^T)^{-1}W \tag{16}$$

Apparently, the optimization problem has two solutions as shown in (15) and (16). Since (15) violates the condition $P > 0$, we accept the solution in (16).

We finally derive P by using (6):

$$AP = P^*e = A(C^T)^{-1}We \tag{17}$$

$$P = (C^T)^{-1}We \tag{18}$$

Since $P^* = (P^*)^T$, we can derive another solution:

$$AP = P^*e = W^TC^{-1}A^Te \tag{19}$$

$$P = A^{-1}W^TC^{-1}A^Te \tag{20}$$

Notice that the result in (18) holds for a single-bottleneck (i.e. single-link) network. In non-vectorized notation, this results translates to:

$$p = \frac{\sum_{f \in F} w_f}{c}$$

The result in (20) holds for a multi-bottleneck network. This result means that each link's optimal price is dependent on the routes of each flow passing through that link. More specifically, the optimal price for link l is accumulation of budgets of flows passing through link l (i.e. W^TA^T in the formula) divided by total capacity of the links that are traversed by the flows traversing the link l (i.e. $A^{-1}C^{-1}$ in the formula). In non-vectorized notation, price of link l can be written as:

$$p_l = \frac{\sum_{f \in F(l)} w_f}{\sum_{f \in F(l)} \sum_{k \in L(f)} c_k}$$

IV. Elasticity

The term *elastic* was first introduced to the networking research community by Shenker [7]. Shenker called applications that adjust their sending rates according to the available bandwidth as “elastic applications”, and the traffic generated by such applications as “elastic traffic”. An example of such traffic is the well-known TCP traffic, which is adjusted according to the congestion indications representing decrease in the available bandwidth. Shenker, further, called applications that do not change their sending rates according to the available bandwidth as “inelastic”. So, this interpretation of *elasticity* is the same as *adaptiveness*, i.e. an application is elastic if it adapts its rate according to the network conditions, it is inelastic if it does not.

The concept of elasticity originates from the theory of economics. In economics, *demand-price elasticity*[2] (i.e. demand elasticity according to price) is defined as *percent change in demand in response to a percent change in price* [8]. In other words, demand elasticity is the responsiveness of the demand to price changes. A formal definition of demand elasticity can be written as [8]:

$$\varepsilon = \frac{\triangle X(p)/X(p)}{\triangle p/p} \tag{21}$$

where p is price, $\triangle p$ is the change in the price, $X(p)$ is user's demand function, and $\triangle X(p)$ is the change in user's demand. (21) can be re-written as:

$$\varepsilon = \frac{p}{X(p)} \frac{dX(p)}{dp} \tag{22}$$

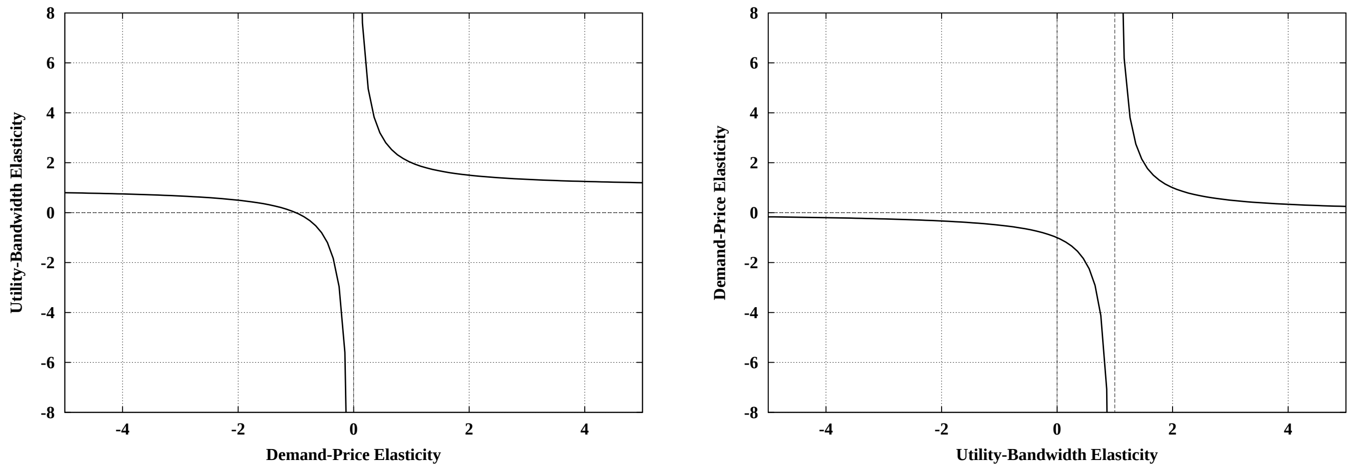
Given ε, elasticity characteristic L_ε of user demand is made according to the following functional definition [8]:

$$L_\varepsilon = \begin{cases} \textit{elastic}, & |\varepsilon| > 1 \\ \textit{unit elastic}, & |\varepsilon| = 1 \\ \textit{inelastic}, & |\varepsilon| < 1 \end{cases}$$

So, Shenker's interpretation of elasticity for user utility is actually different from the real meaning of elasticity in economics. Note that Shenker defined elasticity of user utility according to bandwidth (what we call *utility-bandwidth elasticity*), let's call it ϵ. Let $u(x)$ be user's utility if he is given x amount of bandwidth. Then, following the argument in (22), we can write ϵ as:

$$\epsilon = \frac{x}{u(x)} \frac{du(x)}{dx} \tag{23}$$

[2]Note that demand elasticity can also be defined according to several things other than price (e.g. time of service, delay of service).



(a) Utility-bandwidth elasticity ϵ. (b) Demand-price elasticity ε.

Fig. 1. Utility-bandwidth elasticity ϵ and demand-price elasticity ε with respect to each other.

According to Shenker's interpretation, the functional definition for L_ϵ (i.e. elasticity characteristic of user's utility according to bandwidth) will be as follows:

$$L_\epsilon = \begin{cases} \textit{inelastic}, & \epsilon = 0 \\ \textit{elastic}, & \epsilon \neq 0 \text{ \& user utility is concave} \\ \text{not defined}, & \epsilon \neq 0 \text{ \& user utility is convex} \end{cases}$$

Obviously, L_ϵ is a lot different than L_ε. Basically, L_ε interprets elasticity as *responsiveness* while L_ϵ does it as *adaptiveness.*

We can construct the relationship between ϵ and ε, given that the user solves the well-known maximization problem:

$$\max_x \{u(x) - xp\}$$

The solution to the above problem is $u'(x) = p$. So, given a price p, the user selects his demand such that his marginal utility equals to p. Based on that relationship between the utility function $u(x)$ and the demand function $X(p)$, we can construct the relationship between the demand-price elasticity ε and the utility-bandwidth ϵ elasticity. In the next sub-sections we will formulate the relationship between these elasticities.

A. Utility-Bandwidth Elasticity ϵ

Let $X(p) = Ap^\varepsilon$ where $\varepsilon \neq 0$ and $\varepsilon \neq -1$. Then,

$$p = u'(x) = A^{-1/\varepsilon} x^{1/\varepsilon}$$

$$u(x) = A^{-1/\varepsilon} \left(\frac{1}{\varepsilon} + 1\right) x^{1/\varepsilon + 1}$$

So,

$$\epsilon = \frac{1}{\varepsilon} + 1, \;\; \varepsilon \neq 0 \; \& \; \varepsilon \neq -1$$

Figure 1-a plots ϵ with respect to ε.

B. Demand-Price Elasticity ε

Let $u(x) = Bx^{\epsilon}$ where $\epsilon \neq 1$. Then,

$$u'(x) = p = A\epsilon x^{\epsilon-1}$$

$$X(p) = \left(\frac{p}{A\epsilon}\right)^{\frac{1}{\epsilon-1}}$$

So,

$$\varepsilon = \frac{1}{\epsilon - 1}, \;\; \epsilon \neq 1$$

Figure 1-b plots ε with respect to ϵ.

V. Optimal Prices: Non-Logarithmic Utility Functions

In Section III, we derived optimal prices for the revenue maximization problem $NETWORK(\lambda(p^f))$. In that derivation users demand-price elasticity ε was -1 (see (7)), which means users had *unit elastic* demands. Now, we re-perform the derivation by assuming that users have a utility-bandwidth elasticity of ϵ, where users' demand-price elasticity is $\varepsilon = 1/(\epsilon - 1)$ based on the study in the previous section. Also, *note that* $0 < \epsilon < 1$ *must be satisfied in order to make sure concavity of the utility function.*

First, let B be row vector of the weights that are different for each flow's utility function, and $\hat{B}$ be an $(n \times n)$ matrix in which the element $\hat{B}_{jj}$ is the weight of flow j and all the other elements are zero.

We use a generic utility function. The function and its derivative is as follows:

$$U(\lambda) = B\hat{\lambda}^{\epsilon} \tag{24}$$

$$U'(\lambda) = B\epsilon\hat{\lambda}^{\epsilon-1} \tag{25}$$

According to the relationship between ϵ and ε described in Section IV-A, we can write the demand function and its derivative as follows:

$$\lambda(P^*) = \epsilon^{-\varepsilon} e^T \hat{B}^{-\varepsilon} P^{*\varepsilon} \tag{26}$$

Similarly, we can write derivative of (26) as:

$$\lambda'(P^*) = \epsilon^{-\varepsilon} \varepsilon e^T \hat{B}^{-\varepsilon} P^{*\varepsilon-1} \tag{27}$$

For the revenue maximization problem, we again solve the Lagrangian in (12) but for the new demand function of (26). By plugging (26) and (27) in appropriate places, the optimality conditions for (12) can be written as:

$$L_\gamma : C^T - \epsilon^{-\varepsilon} e^T \hat{B}^{-\varepsilon} P^{*\varepsilon} A = 0 \tag{28}$$

$$L_{P^*} : \epsilon^{-\varepsilon} \varepsilon e^T \hat{B}^{-\varepsilon} P^{*\varepsilon-1}(P^* e - A\gamma) + \epsilon^{-\varepsilon} e^T \hat{B}^{-\varepsilon} P^{*\varepsilon} e = 0 \tag{29}$$

By solving (29) for P^*, we obtain:

$$\varepsilon e^T \hat{B}^{-\varepsilon} P^{*\varepsilon-1}(P^* e - A\gamma) + e^T \hat{B}^{-\varepsilon} P^{*\varepsilon} e = 0 \tag{30}$$

$$P^* = \frac{1}{\epsilon} A\gamma e^{-1} \tag{31}$$

Now, apply (31) into (28) and solve for γ:

$$C^T = \epsilon^{-\varepsilon} e^T \hat{B}^{-\varepsilon} \left(\frac{1}{\epsilon} A\gamma e^{-1} \right)^{\varepsilon} A \tag{32}$$

$$\frac{1}{\epsilon} A\gamma e^{-1} = \epsilon A^{-1/\varepsilon} (C^T)^{1/\varepsilon} (e^T)^{-1/\varepsilon} \hat{B} \tag{33}$$

Substitute (33) into (31) and we obtain P^*:

$$P^* = \epsilon A^{-1/\varepsilon} (C^T)^{1/\varepsilon} (e^T)^{-1/\varepsilon} \hat{B} \tag{34}$$

From (34) we obtain P:

$$AP = P^* e = \epsilon A^{-1/\varepsilon} (C^T)^{1/\varepsilon} (e^T)^{-1/\varepsilon} \hat{B} e \tag{35}$$

$$P = \epsilon A^{-1} A^{|1/\varepsilon|} \left((C^T)^{|1/\varepsilon|} \right)^{-1} (e^T)^{|1/\varepsilon|} \hat{B} e \tag{36}$$

$$P = \epsilon A^{-1} A^{|1/\varepsilon|} \left((C^T)^{|1/\varepsilon|} \right)^{-1} (e^T)^{|1/\varepsilon|} \left(\hat{B}^{|\varepsilon|} \right)^{|1/\varepsilon|} e \tag{37}$$

The result in (36) implies the same thing as in the case of logarithmic utility functions except that the link capacities must be taken more conservatively depending on the elasticity (ϵ or ε by choice) of flows. Observe that as flows demand-price elasticity ε gets higher, the capacity must be taken more conservatively based on the formula $(C^T)^{|1/\varepsilon|}$. Also observe that as flows utility-bandwidth elasticity ϵ gets higher, the capacity must be taken more conservatively based on the formula $(C^T)^{|1/\varepsilon|} = (C^T)^{|\epsilon-1|}$.

Based on (37) we can write the optimal price formulas for single-bottleneck and multi-bottleneck cases respectively as follows in non-vectorized form:

$$p = \epsilon \left(\frac{\sum_{f \in F} w_f^{|\varepsilon|}}{c} \right)^{|1/\varepsilon|}$$

$$p_l = \epsilon \left(\frac{\sum_{f \in F(l)} w_f^{|\varepsilon|}}{\sum_{f \in F(l)} \sum_{k \in L(f)} c_k} \right)^{|1/\varepsilon|}$$

VI. Summary

In this paper, we tried to answer the question of how should user's elasticity to price and bandwidth effect the pricing strategy. We first formulate the optimization problem of total user utility maximization. Based on logarithmic user utility functions, we derived optimal prices.

Then, we revised the term *elasticity* in the area of networking, and defined utility-bandwidth elasticity. We also determined relationship between utility-bandwidth elasticity and the well-known demand-price elasticity. Based on this investigation of elasticity, we then defined a non-logarithmic form of utility functions which include elasticity as a parameter. Considering the newly defined non-logarithmic and concave utility functions, we re-solved the optimization problem of total user utility maximization.

We illustrated that elasticity should take a major role in pricing, since the derived optimal price included elasticity as power of available capacity. This means that pricing strategy should be more conservative in using the available capacity when user's elasticity increases.

References

[1] F. P. Kelly, "Charging and rate control for elastic traffic," *European Transactions on Telecommunications*, vol. 8, pp. 33–37, 1997.

[2] F. P. Kelly, A. K. Maulloo, and D. K. H. Tan, "Rate control in communication networks: Shadow prices, proportional fairness and stability," *Journal of Operations Research Society*, vol. 49, pp. 237–252, 1998.

[3] S. H. Low and D. E. Lapsley, "Optimization flow control – I: Basic algorithm and convergence," *IEEE/ACM Transactions on Networking*, vol. 7, no. 6, pp. 861–875, 1999.

[4] S. Kunniyur and R. Srikant, "End-to-end congestion control: Utility functions, random losses and ECN marks," in *Proceedings of Conference on Computer Communications (INFOCOM)*, 2000.

[5] X. Wang and H. Schulzrinne, "Pricing network resources for adaptive applications in a Differentiated Services network," in *Proceedings of Conference on Computer Communications (INFOCOM)*, 2001.

[6] J. Mo and J. Walrand, "Fair end-to-end window-based congestion control," *IEEE/ACM Transactions on Networking*, vol. 8, no. 5, pp. 556–567, October 2000.

[7] S. Shenker, "Fundamental design issues for the future Internet," *IEEE Journal of Selected Areas in Communications*, vol. 13, pp. 1176–1188, 1995.

[8] H. R. Varian, *Intermediate Microeconomics: A Modern Approach*, W. W. Norton and Company, 1999.