A Small Gain Approach to Delay Robustness of Networks

X. Fan,[*] S. Kalyanaraman, M. Arcak
Department of Electrical, Computer, and Systems Engineering
Rensselaer Polytechnic Institute
Troy, NY 12180

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Abstract

This paper studies the stability of a network flow model with transmission and queuing delays in the forward and backward channels. We present a novel small gain approach to prove global asymptotic stability for arbitrary time delays and network routing. This approach uses a logarithmic state transformation suggested recently in the literature, and establishes a linear input-to-state gain for the transformed system. With the new state variables the gain of the routing matrix is unity and, thus, the stability condition is scalable and independent of routing. Unlike existing results that employ the logarithmic transformation, we give a simple small-gain interpretation for

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[*] Corresponding author. Electrical, Computer and Systems Engineering Department, Rensselaer Polytechnic Institute, Troy, NY 12180-3590, USA. Tel.: +1-518-276-8205; fax: +1-518-276-6261.

the delay robustness of networks. We then demonstrate that the small-gain technique is suitable for the design of a new class of AQM algorithms where the virtual capacity is time-varying, and also can be generalized to other type of networks, such as CDMA uplink power control.

1 Introduction

In this paper, we consider a general topology network consisting of an arbitrary number of sources and links, which are interconnected through the routing matrices, R_f and R_b as in Kelly [1], Low and Lapsley [2], and Kunniyur and Srikant [3].

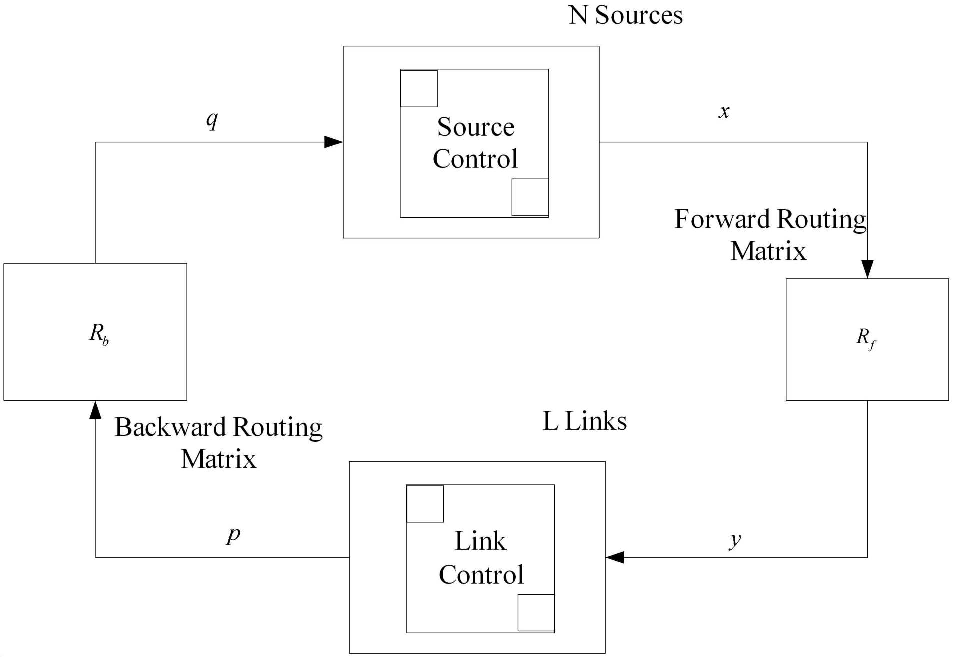

Fig. 1 Network Flow Control Model

Packets from each source (with sending rate x_i) are routed through the links with the aggregate link rate $y = R_f x$, where R_f is the forward routing matrix. Each link l has a fixed capacity c_l, and based on its congestion and queue size, a link price p_j is computed:

$$p_l = h_l\left(y_l, c_l\right), \quad l = 1, \cdots, L. \tag{1}$$

$h_l\left(y_l, c_l\right)$is either chosen as

$$h\left(y_l, c_l\right) = \frac{\max\left\{0, \left(y_l - c_l\right)\right\}}{y_l} \tag{2}$$

to imply the loss probability at link$j$ [3], or

$$h\left(y_l, c_l\right) = \left(\frac{y_l}{c_l}\right)^{\mu}. \tag{3}$$

for a more general case [4]. The link price information is then sent back to each source with the aggregate source price, $q = R_b^T p$, where R_b is the return routing matrix. If the delays are ignored, then $R_f = R_b =: R$ since the links only feed back the price information to the sources that utilize them. In the presence of delays, R_f (respectively, R_b) is obtained from R by multiplying its entries $e^{-\tau_{ij}^f s}$ ($e^{-\tau_{ij}^b s}$), where $e^{-\tau_{ij}^f s}$ is the forward delay from the source i to link j, and $e^{-\tau_{ij}^b s}$ is the backward delay from link j to source i.

Sources use the aggregate price q to update their sending rates. In the paper, we consider the following TCP-like congestion control algorithm, which is a generalization of the one suggested in [4]

$$\dot{x}_i\left(t\right) = \kappa_i g_i\left(x_i\left(t\right), x_i\left(t - T_i\right)\right)\left(\frac{1}{x_i^{\alpha}\left(t\right)} - q_i\left(t\right)\right), \quad i = 1, \cdots, N. \tag{4}$$

where $T_i = \tau_{ij}^f + \tau_{ij}^b$ denotes the round trip delays for source i. Our goal is to derive conditions, under which of the network flow model (1)-(4) is globally asymptotically stable for arbitrary forward and backward delays and network topology. Our main interest in this formulation, rather than a specific protocol, is that encompasses numerous protocols such as variant of TCP Reno etc. Transmission and queuing delays can be significant in these networks, and threaten stability properties achievable for delay-free models. The class (1)-(4) also encompasses other applications, such as power control in CDMA, control systems where feedback signals are generated at remote locations, and transmitted to the actuators via communication channels [5].

In Part One of the paper, we present a novel small gain approach to prove global asymptotic stability for system (1)-(4) to remain globally asymptotically stable for arbitrary time delays and network routing. The main idea is to use a logarithmic state transformation suggested recently in the literature [6], and establishes a linear input-to-state gain [7] for the transformed system. With the new state variables the gain of the routing matrix is unity and, thus, the stability condition is scalable and independent of routing. Unlike existing results that employ the logarithmic transformation, we give a simple small-gain interpretation for the delay robustness of networks.

In Part Two of the paper we employ the small-gain technique to design a new class of AQM algorithms where the virtual capacity is time-varying. With the help of the logarithmic state transformation, we can regard the variation of the virtual capacity as an additive disturbance and show that the transformed states of the nominal system satisfies an input-to-state stability (ISS) gain with respect to this disturbance [7]. We then proceed to derive conditions under which the same disturbance satisfies a complementary ISS gain with respect to the transformed states.

We conclude the paper with the extension of this small-gain technique to delay robustness of a gradient algorithm of CDMA uplink power control proposed in [8, 9]. Compared with the study in [10] under a passivity framework, the result in this paper is delay independent and does not rely on the channel gain.

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